The Holography Mistake: Why ’t Hooft’s “Holographic Principle” Is a Conjecture—and How to State It in Pure Mathematics

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Many readers assume the “Holographic Principle” is already a theorem with a settled mathematical proof, later “applied” to dramatic cases. It isn’t. ’t Hooft’s formulation is a conjectural organizing idea: that a complete description of a higher-dimensional domain is, in principle, encodable by data on a lower-dimensional boundary. Powerful as it is, the principle lacks a single canonical formal statement and a model-independent proof. What we *do* possess are (i) rigorously proved theorems in mathematics that realize “boundary determines bulk” in specific settings (inverse problems, elliptic PDEs, Radon transforms, dualities like Gelfand and Tannaka–Krein), and (ii) precise, model-dependent equivalences in specialized frameworks. This paper has two aims: first, to make the status claim explicit—conjecture, not theorem—by clarifying what a theorem would have to look like; second, to supply a pure-mathematics template (“Mathematical Holography”) that captures the structural content: categories of bulks and boundaries, a boundary functor, observable functors, locality (sheaf/cosheaf), and reconstruction/equivalence criteria. We then map well-known mathematical results into this template, identify admissibility hypotheses and failure modes, and offer a checklist for future theorems. The goal is conceptual hygiene and constructive progress: replace vague lore with crisp statements you can actually prove—or falsify.

Keywords: holographic principle, conjecture not theorem, boundary encodes bulk, reconstruction, inverse problems, Calderón problem, elliptic PDE, Dirichlet to Neumann map, Radon transform, Gelfand duality, Tannaka–Krein duality, sheaf and cosheaf, categorical equivalence, admissibility conditions, uniqueness and stability, mathematical template, failure modes, pedagogy.

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References

1. Premise and Claim

1.1 The common assumption: "It's already a theorem"

In much writing, "the holographic principle" is treated as if it were a settled mathematical fact: a precise statement with agreed hypotheses and an accepted proof. The impression arises because (i) the principle is often phrased with theorem-like certainty; (ii) famous special cases and powerful heuristics are cited as if they constituted a general proof; and (iii) survey narratives compress a long chain of ideas into a single "result." Readers infer: there exists a formal statement P with clearly stated assumptions H such that P follows from H in a standard axiomatic framework. But that inference is incorrect. What exists are multiple informal formulations that rhyme with one another (boundary encodes bulk; area-like bounds; reconstruction from boundary data), plus rigorous theorems that establish *pieces* of this story in *restricted* settings. The jump from "compelling pattern with many confirmed fragments" to "general theorem" is where the assumption goes wrong.

1.2 Our thesis: conjecture, not theorem — what this means in math standards

By mathematical standards, calling something a *theorem* requires:

- a fixed ambient framework (e.g., category, topology, measure, algebra) and a common vocabulary,
- a precise statement with quantifiers, hypotheses, and claimed conclusions,
- a proof accepted by the community that derives the conclusion from the hypotheses using permitted inferences.

The "holographic principle," as generally invoked, fails this test on two fronts:

(1) **Lack of a canonical formalization.** There is no single, community-standard statement T such that everyone agrees on its objects, morphisms, and hypotheses. Instead there are families of statements of the form "bulk structure is encoded by boundary data," which are not mutually equivalent and live in different mathematical habitats.

(2) **Absence of a general proof.** Where precise statements *do* exist (inverse problems, tomography, duality theorems, categorical reconstructions), the theorems either prove weaker claims (e.g., unique determination of one structure by another) or prove "holography-like" results under strong additional structure. None of these constitutes a general proof of a universal bulk–boundary equivalence.

Thus, in math terms, "the holographic principle" is a *conjectural organizing heuristic*: a program that suggests what to try to formalize and prove. Calling it a conjecture is not a demotion; it is

conceptual hygiene. It keeps us honest about which parts are proved, which depend on extra hypotheses, and where the open problems really are.

1.3 What would count as a proof (objects, axioms, conclusion)

To convert "holography" into a bona fide theorem, one needs an explicit triple (C, D, ∂) of mathematical data, plus functorial "observables," and a target equivalence. The minimal scaffold looks like this:

Objects and categories.

- Let C be a category of "bulks" (e.g., a class of spaces with structure, operator-algebraic systems, schemes with extra data).
- Let D be a category of "boundaries" (e.g., boundary spaces/divisors, representation categories, boundary algebras).
- A functor ∂ : C -> D assigns to each bulk its boundary object, functorially on morphisms.

Observable functors.

- Functors F : C -> Alg and G : D -> Alg that assign algebraic data (algebras, modules, states, kernels) to bulks and boundaries, respectively.
- Locality: F (and/or G) should satisfy sheaf/cosheaf axioms over suitable sites so that global data are reconstructed from compatible local pieces.

Axioms (examples).

- Regularity/rigidity: unique continuation, maximum principles, or duality properties ensuring that interior data are pinned by boundary data.
- Exactness: F and G preserve limits/colimits needed for gluing and refinement (e.g., compatibility with covers and inverse limits).
- Richness/separation: G(∂X) must separate non-isomorphic bulks X; i.e., the boundary observables distinguish different interiors inside the admissible class.
- Admissibility: restrict to a subcategory C_adm where pathologies are excluded (e.g., analytic coefficients, nontrapping geometry, finite-type conditions).

Conclusions to prove (sample targets).
There are several escalating notions of success; any one of them, stated precisely and proved, would qualify as a "theorem of holography" in that setting.

- **Isomorphism of observables (weak reconstruction).**
  Prove a natural isomorphism

$$F(X) \cong G(\partial X)$$

for all X in C_adm, compatible with restriction/gluing. This shows that the interior's observable algebra equals the boundary's observable algebra.

- **Faithfulness / injectivity (distinguishability).**
  If ∂X_1 \cong ∂X_2 and G(∂X_1) \cong G(∂X_2), then X_1 \cong X_2 in C_adm. This says the boundary data determine the bulk up to isomorphism.
- **Equivalence of categories (strong reconstruction).**
  Exhibit a functor R : D_adm -> C_adm such that

$$\partial \circ R \cong \mathrm{id}_{D_{\mathrm{adm}}}, F \cong G \circ \partial,$$

and show that ∂ is an equivalence between C_adm and a reflective (or coreflective) subcategory of D_adm. This yields true bulk-from-boundary reconstruction.

- **Monoidal and limit compatibility.**
  Check that the isomorphisms are natural in disjoint unions/products and commute with the limits used to approximate bulks by boundary refinements.

Proof obligations.

- Specify the exact hypotheses (regularity class, topology, algebraic finiteness, etc.).
- Construct the comparison maps (e.g., Poisson-type operators, Radon inverses, Tannaka reconstruction).
- Prove: (i) well-definedness and functoriality, (ii) locality/gluing, (iii) injectivity/surjectivity (or a description of the kernel), and (iv) stability under refinement.

If such a package is delivered in a concrete setting, then one has a *theorem-level* holography statement there. Short of that, the general "boundary encodes bulk" slogan remains a *conjecture* or organizing principle, and should be labeled as such.

2. A Pure-Math Template: “Mathematical Holography (MH)”

2.1 Bulk category C, boundary category D, boundary functor \partial : C -> D

Objects.

- C: a category of “interiors/bulks.” Examples: compact manifolds with boundary; finitely generated C*-systems with a designated subalgebra; algebraic varieties with a normal crossings boundary divisor; metric spaces with a chosen corona.
- D: a category of “boundaries.” Examples: smooth boundary manifolds; boundary algebras; boundary divisors; Cech coronas.

Morphisms.

- C-morphisms should preserve the interior structure and map boundary to boundary.
- D-morphisms are the induced boundary maps.

Boundary functor.

- \partial : C -> D is functorial: for f : X -> Y in C, there is \partial f : \partial X -> \partial Y in D, with \partial(id_X)=id_{\partial X} and \partial(g\circ f)=\partial g \circ \partial f.

Admissible subcategory.

- Often we restrict to C_adm subset C where regularity, finiteness, or rigidity hold (e.g., analytic coefficients, compactness, nondegeneracy), because reconstruction typically fails without hypotheses.

2.2 Observable functors F : C -> Alg and G : D -> Alg

Idea. Encode each object by an algebraic “shadow” that we can compare across bulk and boundary.

Targets.

- Alg may be a category of algebras (commutative rings, C*-algebras, von Neumann algebras), modules, or structured vector spaces (Hilbert spaces, kernels, distributions).

Examples.

- Topological/smooth: F(X)=algebra of interior solutions (e.g., harmonic functions, eigenspaces, de Rham cohomology), G(\partial X)=boundary traces, boundary cohomology, or Dirichlet-to-Neumann operators-as-algebras.

- Algebraic: F(X)=QCoh(X) or D^bCoh(X); G(\partial X)=QCoh(\partial X) with extra gluing data.
- Operator-algebraic: F(X)=A_X (interior observable algebra), G(\partial X)=B_{\partial X} (boundary algebra).

Natural transformations.

- We seek a comparison map \eta_X : F(X) -> G(\partial X) (trace, restriction, Poisson operator, Radon transform, boundary representation) that is natural in X.

2.3 Locality: sheaf/cosheaf conditions and gluing

Sheaf (restriction) axiom for F.
Let {U_i} be a cover of X in a site on C (e.g., open cover, Zariski cover). Then F is a sheaf if

- (S1) For each i, restriction r_i : F(X) -> F(U_i).
- (S2) If s_i in F(U_i) agree on overlaps (r_{ij}(s_i)=r_{ji}(s_j) in F(U_i \cap U_j)), then there exists a unique s in F(X) with r_i(s)=s_i for all i.

Cosheaf (extension) axiom for G on boundaries.
Given a boundary cover {\partial U_i} of \partial X,

- (C1) Co-restrictions q_i : G(\partial U_i) -> G(\partial X).
- (C2) If data t_i in G(\partial U_i) are compatible on overlaps, they glue to a unique t in G(\partial X).

Compatibility of \eta with gluing.
For every cover, the diagram

```
F(X)  --\eta_X-->  G(\partial X)
 |                  |
 v                  v
lim F(U_i)  -->>  lim G(\partial U_i)
```

commutes (limits = equalizers for sheaves; colimits for cosheaves). Concretely: “restrict then compare” = “compare then glue.”

Why locality matters.
Reconstruction is rarely global at once; it proceeds by local pieces that glue. Sheaf/cosheaf conditions ensure that if comparison holds locally and compatibly, it lifts globally.

2.4 Reconstruction targets: F(X) ~= G(\partial X); or equivalence via a right inverse R to \partial

We state success goals in increasing strength.

(T1) Isomorphism of observables (weak reconstruction).
There exists a natural isomorphism

- \eta_X : F(X) \xrightarrow{\sim} G(\partial X),
  natural in X and compatible with sheaf/cosheaf gluing.
  Interpretation: “All interior observables equal boundary observables.”

(T2) Boundary sufficiency (faithfulness/injectivity).
If \partial X_1 \cong \partial X_2 in D and G(\partial X_1) \cong G(\partial X_2) as Alg, then X_1 \cong X_2 in C_adm.
Interpretation: “Boundary data distinguish interiors.”

(T3) Right inverse and essential surjectivity (strong reconstruction).
There exists R : D_adm -> C_adm with \partial \circ R \cong id_{D_adm} and a natural isomorphism F \cong G \circ \partial.
If additionally every X in C_adm is isomorphic to R(\partial X) (or more precisely, the unit/counit are isomorphisms), then \partial is an equivalence of categories between C_adm and a reflective/coreflective subcategory of D_adm.
Interpretation: “Every admissible boundary arises from a bulk, and the bulk is (essentially) unique.”

(T4) Stability/robustness.
The comparison respects perturbations/covers: small changes in boundary data produce controlled changes in reconstructed interior data (continuity or Lipschitz-type bounds in a chosen topology on Alg).

2.5 Monoidal/limit exactness and refinement stability

Monoidal structure.
Suppose (C, \sqcup) and (D, \sqcup) admit disjoint union (or another monoidal product), and Alg has \otimes. We require:

- \partial(X \sqcup Y) \cong \partial X \sqcup \partial Y,
- F(X \sqcup Y) \cong F(X) \otimes F(Y),
- G(\partial X \sqcup \partial Y) \cong G(\partial X) \otimes G(\partial Y),

- \eta_{X \sqcup Y} = \eta_X \otimes \eta_Y (naturality under monoidal product).

Limits/colimits.
For inverse systems of refinements {X_\alpha} with X = lim X_\alpha (e.g., nested exhaustions; boundary meshes getting finer):

- Exactness: F(lim X_\alpha) \cong lim F(X_\alpha).
  Similarly, for boundaries and G with colimits if relevant.
  This ensures reconstruction commutes with approximation/refinement.

Refinement stability.
Given a directed family of boundary refinements {\partial X_\alpha} whose colimit is \partial X, we require the comparison maps to be compatible:

- colim G(\partial X_\alpha) \xrightarrow{\sim} G(\partial X),
- and under \eta, colim F(X_\alpha) \xrightarrow{\sim} F(X).

Why exactness/stability matter.
Most reconstructions are obtained by: discretize the boundary -> compute boundary observables -> pass to the limit. If exactness fails, the limit may not match the true interior data; exactness guarantees that “refine then compare” = “compare then refine.”

Minimal checklist for proving MH in a concrete setting

- Pick C, D, \partial with a clear admissible class.
- Define F and G so that: (i) they are sheaves/cosheaves on natural sites; (ii) they have the needed monoidal and limit exactness.
- Construct \eta_X and prove it is natural and local (compatible with covers).
- Establish one of (T1)–(T3) and, if possible, stability (T4).
- Record failure modes (where admissibility is essential): nonuniqueness, loss of stability, failure of exactness under limits.

This template is entirely math-internal: no physics assumptions are needed. It turns the slogan “boundary encodes bulk” into precise, testable statements you can actually prove (or falsify) in your chosen category.

3. Worked Mathematical Realizations (No Physics)

3.1 Elliptic PDE: Dirichlet uniqueness; Poisson operator reconstruction

Setting. Let (M,g) be a compact C^\infty Riemannian manifold with nonempty boundary dM. Consider the Laplace–Beltrami operator Delta.
Dirichlet problem. Given phi in C^\infty(dM), find u with

- Delta u = 0 in M,
- u|*{dM} = phi.*
  Under standard compatibility (none needed here) and regularity of (M,g), there exists a unique u in C^\infty(M); write u = P(phi), where P is the ***Poisson operator*** *P : C^\infty(dM) -> C^\infty(M).*
  Uniqueness (Dirichlet). *If u is harmonic with u*|{dM}=0, then u=0. Hence the **trace** map tr : Harm(M) -> C^\infty(dM) is injective and P is its inverse on the image.
  MH mapping. Take
- C = objects (M,g) with Harm(M) as "interior observables" F(M)=Harm(M),
- D = dM with boundary data G(dM)=C^\infty(dM),
- eta_M : F(M) -> G(dM) given by restriction, with inverse P.
  Then **F(M) ~= G(dM)** naturally; locality holds via sheaf properties of harmonic functions. This is a clean instance of MH-(T1) with explicit reconstruction.

Stability. In Sobolev scales, P : H^{s-1/2}(dM) -> H^{s}(M) is bounded for s>1/2; dependence on phi is continuous (elliptic regularity).

3.2 Calderón's inverse problem: Lambda_gamma determines gamma (hypotheses, uniqueness/stability)

Setting. Let Omega subset R^n be bounded with C^{1,1} boundary. A (scalar) conductivity gamma(x) satisfies 0 < c <= gamma in L^\infty(Omega). Define the **Dirichlet-to-Neumann (DtN) map**

- Lambda_gamma : H^{1/2}(dOmega) -> H^{-1/2}(dOmega),
- Lambda_gamma(phi) = gamma * d_n u|*{dOmega}, where u solves div(gamma grad u)=0 in Omega, u*|{dOmega}=phi.

Inverse question. Does Lambda_gamma determine gamma?

Uniqueness (representative results).

- n >= 3: If gamma in C^{2} (or various W^{1,n} \cap L^\infty refinements), then **gamma is uniquely determined by Lambda_gamma** (Sylvester–Uhlmann and successors).
- n = 2: For gamma in L^\infty with positive bounds, **uniqueness holds** (Astala–Päivärinta); smoother classes admit constructive reconstructions (Nachman).
- **Anisotropic** case: uniqueness up to boundary-fixing diffeomorphisms (gauge) — i.e., one recovers the equivalence class of tensors.

Stability. Generally **logarithmic-type** instability (Alessandrini): small boundary errors can cause large interior variations; Hölder stability holds under stronger a priori constraints (e.g., piecewise analytic/polyhedral classes).

MH reading. C = admissible conductivities; D = boundary operators; F = id on gamma, G = id on Lambda_gamma; eta : gamma |-> Lambda_gamma is **injective** on admissible classes (MH-(T2)); in some constructive regimes, a (possibly discontinuous) right-inverse gives **reconstruction**.

3.3 Radon transforms: injectivity and inversion on suitable function spaces

Setting. For f on R^n with suitable decay/smoothness, the **Radon transform** Rf maps f to its integrals over affine hyperplanes:

- Rf(theta,s) = integral_{x·theta = s} f(x) d sigma(x), with theta in S^{n-1}, s in R.

Injectivity and inversion. On Schwartz functions (and many Sobolev/L^2 scales), R is injective and admits explicit inversion:

- f(x) = c_n * (-Delta)^{(n-1)/2} (R^* R f)(x),
  where R^* is the backprojection and the fractional power is defined via Fourier multipliers (odd n) or via filtered backprojection (even n).
  Limited-angle and support theorems give further boundary-like reconstruction statements under geometric coverage conditions.

MH reading. C = function spaces on R^n (bulks); D = data on the lower-dimensional family of hyperplanes (boundaries); F = identity on functions, G = Radon data; eta = R is injective with an explicit inverse on admissible spaces (MH-(T3) in that setting). Monoidal additivity and limit exactness track standard distribution-space functoriality.

3.4 Gelfand duality: compact Hausdorff spaces <-> commutative C*-algebras

Statement. The contravariant functor X |-> C(X) gives an **equivalence of categories**

- CompHaus^{op} ~= CommCstar,
  between compact Hausdorff spaces (with continuous maps) and commutative unital C*-algebras (with *-homomorphisms). The inverse sends A to its **Gelfand spectrum** Spec(A).

Reconstruction. The "space" X is completely recovered from the algebra C(X); conversely, the algebra of complex-valued continuous functions on X is recovered from X.

MH reading. C = compact Hausdorff spaces, D = commutative C*-algebras, \partial = C(·) (a boundary-like externalization into observables). Then F = id on C, G = id on D, eta_X : X <-> Spec(C(X)) is a contravariant **equivalence** (a very strong MH-(T3) paradigm: interior object equals its "observable boundary" algebra).

3.5 Tannaka–Krein: groups reconstructed from representation categories

Setting. Let G be a compact (or more generally, suitable) topological group. Consider its **tensor category of finite-dimensional continuous unitary representations** Rep(G), together with the forgetful (fiber) functor omega : Rep(G) -> Vec.

Tannaka–Krein duality. The pair (Rep(G), omega) determines G up to isomorphism; conversely, every suitable rigid symmetric monoidal category with a fiber functor arises as Rep(G). More abstractly, G = Aut^{\otimes}(omega).

MH reading. C = compact groups, D = rigid symmetric monoidal categories with a fiber functor, \partial(G) = (Rep(G), omega). The reconstruction functor R realizes **G from its boundary representation data**. This is an explicit MH-(T3): equivalence between a "bulk" object and its "boundary" probing data.

3.6 Conformal welding / Riemann mapping: boundary data fixes interior up to normalization

Riemann mapping (simply connected). If U subset C is simply connected and not all of C, there exists a conformal bijection f : U -> D (unit disk), unique up to post-composition by an automorphism of D. With a boundary parametrization in appropriate classes (e.g., Jordan curves), f extends to a homeomorphism of closures; the **boundary correspondence** essentially fixes f up to the 3-parameter automorphism group.

Conformal welding. Given a (quasi)symmetric homeomorphism h : S^1 -> S^1, there exist conformal maps f_i from the inside/outside of S^1 to complementary domains whose boundary

values satisfy f_+ = f_- o h on S^1. The pair of domains is **reconstructed** from h, uniquely up to post-composition by a Möbius transformation.

MH reading. C = planar domains up to conformal equivalence, D = boundary homeomorphisms/data; \partial takes a domain to its boundary identification. The welding theorem gives **reconstruction from boundary gluing data** (MH-(T3) up to the natural normalization). Exactness under refinement corresponds to standard Carathéodory convergence and continuity of boundary correspondences.

Takeaway. Each of 3.1–3.6 is a **proven** "boundary determines bulk" theorem in its own habitat. None claims a universal bulk–boundary equivalence across all mathematics; instead, each instantiates the MH template with explicit objects, functors, locality, and reconstruction—precisely the standard one needs when upgrading a slogan to mathematics.

4. Admissibility & Axioms

4.1 Regularity, unique continuation, analyticity/ellipticity

Purpose: prevent "hidden interiors" that boundary data cannot control.

- Ellipticity (uniform). For PDE settings, assume a uniformly elliptic operator L with bounded coefficients. This yields existence, uniqueness, and interior regularity for boundary-value problems.
- Unique continuation (UCP). If L u = 0 and u vanishes to infinite order at the boundary (or on a set with an accumulation point), then u = 0. Tools include Carleman estimates and Holmgren-type results.
- Analyticity vs finite smoothness. Analytic or Gevrey coefficients give the strongest continuation and Runge approximation. With only L^infty coefficients, UCP may fail; admissibility should impose higher regularity (e.g., W^{1,n} intersect L^infty or C^{1,1}) or structured piecewise-analytic classes.
- Geometric regularity of the boundary. C^{1,1} (or better) boundaries ensure well-posed traces, Poisson operators, and Dirichlet-to-Neumann maps. Corners require weighted function spaces.

MH role: Ellipticity + UCP justify injectivity of trace-type functors, existence of Poisson or extension operators, and uniqueness in inverse problems, enabling MH targets T1 and T2.

4.2 Separation richness of boundary observables

Purpose: boundary data must distinguish non-isomorphic interiors.

- Separating families. A boundary observable functor G is separating on an admissible class if
  G(partial X1) isomorphic to G(partial X2) implies X1 isomorphic to X2.
  Examples:
  1. Full Dirichlet-to-Neumann map for admissible conductivities.
  2. Full Radon data over all hyperplanes.
  3. C(X) in Gelfand duality.
  4. Rep(G) with tensor structure and a fiber functor in Tannaka–Krein.
- Sufficiency vs insufficiency.
  Sufficient: complete boundary operators or spectra; full angular coverage in tomography; full tensor data in representation categories.
  Insufficient: a single scalar summary (total flux), finitely many moments, limited-angle Radon data, boundary values without normal derivatives for elliptic inverse problems.
- Minimality. Often a minimal separating set exists (for example, all pairwise boundary voltage responses; a generating set of irreps with tensor rules). Admissibility should specify the required breadth of boundary probes.

MH role: separation is the content of MH target T2. Without it, many interiors share the same boundary "shadow."

4.3 Exactness under covers, limits, and monoidal sums

Purpose: make reconstruction commute with how objects are built in practice.

- Sheaf exactness (covers). For an open cover {U_i} of X, a sheaf F satisfies the equalizer sequence
  0 -> F(X) -> product_i F(U_i) ⇒ product_{i,j} F(U_i intersect U_j).
  This guarantees interior data are determined by compatible local pieces. On the boundary, a cosheaf condition handles extension and gluing of G(partial U_i).
- Directed limits (refinements). For a refinement system {X_alpha} with X = lim X_alpha, require
  F(X) isomorphic to lim F(X_alpha), and G(partial X) isomorphic to colim G(partial

X_alpha).
This may need compactness, nuclearity, or Mittag–Leffler-type conditions to avoid lim^1 pathologies.

- Monoidal additivity. For disjoint unions or products:
  partial(X ⊔ Y) isomorphic to partial X ⊔ partial Y,
  F(X ⊔ Y) isomorphic to F(X) tensor F(Y),
  G(partial X ⊔ partial Y) isomorphic to G(partial X) tensor G(partial Y),
  and the comparison maps satisfy eta_{X ⊔ Y} = eta_X tensor eta_Y.

- Continuity and stability. Small boundary perturbations should not create wild interior changes; require continuity (or quantitative bounds) of reconstruction in chosen norms or topologies on Alg.

MH role: exactness guarantees “refine or glue at the boundary, then reconstruct” equals “reconstruct, then refine or glue.”

4.4 When and why admissibility is necessary (counterexamples)

Purpose: show what fails without the axioms so admissibility is seen as sharp, not cosmetic.

- Failure of UCP (rough coefficients). With only L^infty leading coefficients, second-order elliptic equations can admit nontrivial solutions that vanish on a set yet are not identically zero. Boundary traces then do not determine interior solutions; MH target T1 fails.

- Anisotropic nonuniqueness (gauge). In tensor conductivity, the Dirichlet-to-Neumann map is invariant under boundary-fixing diffeomorphisms. Without gauge fixing (choosing a representative), separation fails.

- Limited-angle tomography. Radon data restricted to a cone of directions do not determine f uniquely; whole wavefront sets can be invisible. Admissibility must require full angular coverage or microlocal conditions.

- Pathological boundaries and topology. Highly irregular or fractal boundaries can break trace theorems; Poisson operators need not be bounded; conformal welding can be nonunique or ill-posed without quasisymmetric regularity.

- Inverse-limit pathology. Without ML-type conditions, F(lim X_alpha) isomorphic to lim F(X_alpha) can fail; reconstruction by boundary refinement may not converge to the true interior invariant.

- Insufficient boundary observables. Using only a scalar boundary functional identifies many non-isomorphic interiors; separation richness is violated.
- Representation collapse. In Tannaka settings, forgetting tensor structure (keeping only the character table or Grothendieck ring) need not reconstruct the group; monoidal data and a fiber functor are essential.

MH lesson: each failure points to a needed axiom — regularity and UCP, gauge fixing, data richness, geometric tameness, exactness under limits, or stability. A credible MH theorem states these conditions up front; outside them, "boundary encodes bulk" is false or not useful.

5. Failure Modes and Counterexamples

5.1 Non-uniqueness: indistinguishable boundaries for distinct interiors

- Anisotropic conductivities with a gauge: in the tensor version of the Calderón problem, pushing forward the metric or conductivity by a diffeomorphism that fixes the boundary leaves the Dirichlet to Neumann map unchanged. Distinct interiors share the same boundary operator unless a gauge is fixed.
- Limited observations: in tomography with a restricted angular set, different interior functions can produce identical limited angle data. Microlocally, entire wavefront sets can be invisible to the chosen directions.
- Inadequate observables: replacing a full boundary operator by a few scalar summaries (for example a finite set of moments) identifies many nonisomorphic interiors. The boundary data are not separating.
- Representation loss: in Tannaka style reconstruction, keeping only the character table or the Grothendieck ring can fail to distinguish nonisomorphic groups. The tensor structure and a fiber functor are essential.

Moral: without a separating family of boundary observables and the right equivalence relation on interiors, uniqueness fails.

5.2 Instability: severe ill posedness under noise

- Calderón stability: even when uniqueness holds, small boundary perturbations can cause large interior errors. Quantitative results are often logarithmic; reconstruction is exponentially sensitive to noise.

- Limited angle tomography: inversion amplifies high frequencies that are weakly seen by the data, so numerical reconstructions are unstable unless strong regularization is imposed.
- Conformal welding with rough data: welding maps may depend discontinuously on boundary homeomorphisms that are far from quasisymmetric; tiny perturbations can produce large geometric changes.

Moral: an MH statement that ignores stability is of limited practical value. Robustness hypotheses and explicit stability estimates should be part of the theorem.

5.3 Pathologies from low regularity or wild topology

- Failure of unique continuation: for second order elliptic operators with very rough coefficients, nontrivial solutions can vanish on large sets. Boundary traces then do not determine the interior solution space.
- Bad boundaries: fractal or highly nonrectifiable boundaries can break trace theorems and the continuity of Poisson operators; standard Dirichlet theory may fail without weighted spaces and extra assumptions.
- Welding nonuniqueness: without quasisymmetric control, conformal welding can be nonunique or ill posed; the same boundary identification may correspond to multiple noncongruent interiors.
- Limit failures: inverse limits can misbehave without compactness or Mittag Leffler conditions. Equalities such as F of limit equals limit of F can fail, so reconstruction by refinement may not converge.

Moral: regularity and geometric tameness are not cosmetic. They are the line between true reconstruction and false slogans.

5.4 Lessons for stating MH theorems narrowly and honestly

1. Specify the admissible class up front
   Regularity of coefficients, boundary smoothness, geometric constraints, and allowed topologies. Name the gauge and the equivalence relation on interiors.
2. Demand separating boundary data
   State exactly which boundary observables are used and prove that they distinguish admissible interiors. If minimal data are unknown, say so.

3. Include stability, not just uniqueness
   Provide norms and quantitative estimates. If only logarithmic stability holds, declare it and state the implications for approximation and noise.

4. Build in locality and exactness
   Prove sheaf or cosheaf properties, and that reconstruction commutes with covers and limits. Record any failure of exactness and how you avoid it.

5. Document failure modes
   Give explicit counterexamples just outside your hypotheses. Explain why each hypothesis is necessary and which ones are likely artifacts of technique.

6. Calibrate the target
   Be clear whether you prove isomorphism of observable algebras, injectivity of the boundary functor, or a full equivalence of categories. Do not oversell a weak target as a strong one.

Bottom line: a credible MH theorem is narrow by design. It declares the class, the data, the topology, the stability, and the exact sense of reconstruction. Everything else belongs either in examples, counterexamples, or future work.

6. From Template to Theorem: A Minimal MH Prototype

6.1 Statement for harmonic functions on compact manifolds with boundary

Setting. Let M be a compact smooth Riemannian manifold with nonempty boundary, written boundary M. Let Delta be the Laplace–Beltrami operator. Write Harm(M) for the space of smooth functions u on M with Delta u = 0. Write C infinity(boundary M) for smooth boundary data, and H^s for Sobolev spaces.

Theorem (MH prototype, Dirichlet case).
For every s greater than 1 over 2 there exists a bounded linear operator P: H^{s - 1 over 2}(boundary M) -> H^{s}(M) such that

1. (existence) for every phi in H^{s - 1 over 2}(boundary M), the function u = P(phi) satisfies Delta u = 0 in M and u on boundary M equals phi in the trace sense;

2. (uniqueness) if u in H^{1}(M) satisfies Delta u = 0 and u on boundary M equals 0, then u equals 0;

3. (inverse to trace) the trace map T: Harm(M) -> C infinity(boundary M), T(u) = u on boundary M, is bijective onto C infinity(boundary M), with inverse P restricted to smooth data;

4. (natural isomorphism of observables) taking F(M) = Harm(M) and G(boundary M) = C infinity(boundary M), the comparison map eta_M = T gives a natural isomorphism F(M) isomorphic to G(boundary M);

5. (locality) the assignment U mapsto Harm(U) is a sheaf on M with respect to open covers, and the family of comparison maps is compatible with restriction and gluing along covers of M and along induced covers of boundary M.

This realizes MH target T1 in a fully explicit setting.

6.2 Proof sketch (existence, uniqueness, sheaf compatibility)

Existence. Solve the weak Dirichlet problem via the standard variational method. Choose an extension Phi in H^{1}(M) with Phi on boundary M equal to the given boundary datum phi. Let v minimize the Dirichlet energy integral over M of absolute grad w squared among w in H^{1}_0(M). Set u = Phi minus v. Then u in H^{1}(M) satisfies Delta u = 0 weakly and u on boundary M equals phi. Elliptic regularity upgrades u to H^{s}(M) when phi is in H^{s - 1 over 2}(boundary M), and to smooth when phi is smooth. This defines the Poisson operator P and shows it is bounded H^{s - 1 over 2} to H^{s}.

Uniqueness. If u in H^{1}(M) is harmonic with zero boundary trace, integrate by parts to obtain integral over M of absolute grad u squared equals 0, hence grad u equals 0 and u is constant. The boundary condition forces the constant to be 0. For smooth data the same follows from the maximum principle.

Inverse to trace. For smooth phi, the constructed u = P(phi) is smooth and harmonic with the correct boundary values, so T(P(phi)) = phi. Conversely, for u harmonic and smooth, T(u) is smooth and P(T(u)) solves the same boundary problem as u, hence P(T(u)) = u by uniqueness. Thus T is a bijection between Harm(M) and C infinity(boundary M).

Sheaf compatibility. The presheaf U mapsto Harm(U) satisfies the sheaf equalizer axiom: if {U_i} covers X subset M, and u_i in Harm(U_i) agree on all overlaps U_i intersect U_j, then a unique u in Harm(X) exists with u on U_i equal to u_i. Proof: define u locally by the u_i; uniqueness follows from the uniqueness part above applied to differences with zero boundary data on overlaps; existence uses standard partition and elliptic regularity arguments to pass from local solutions to a global solution consistent on overlaps. The same locality holds for boundary

traces and for Poisson operators on subdomains with induced boundary data. Therefore eta commutes with restriction and gluing.

Conclusion. All items 1 to 5 hold, giving a concrete, complete MH theorem in this setting.

6.3 How to generalize: operators beyond Laplacian, manifolds with corners, anisotropy

A) Uniformly elliptic operators beyond Delta.
Let L u = divergence(A grad u) plus b dot grad u plus c u on M, where A is symmetric and uniformly positive definite, b and c are bounded, and boundary M is at least C^{1,1}. The Dirichlet problem L u = 0 with u on boundary M equal to phi is well posed in H^{1} and in Sobolev scales H^{s}. Existence follows from Lax–Milgram or Fredholm theory; uniqueness from coercivity or the maximum principle; elliptic regularity yields the same mapping properties P: H^{s - 1 over 2}(boundary M) -> H^{s}(M). Under these hypotheses, the same MH targets hold with F(M) the solution space of L u = 0.

B) Manifolds with corners and piecewise smooth boundary.
Replace standard Sobolev spaces by weighted or adapted scales (for example b–Sobolev spaces). The trace and Poisson operators exist on each boundary face, together with compatibility conditions on corners. The sheaf property still holds when open sets respect the stratification. The MH statement remains valid after explicitly formulating the boundary category to include the face structure and matching conditions.

C) Anisotropy.
Let the operator be divergence of Gamma grad u with Gamma a positive definite matrix field (a conductivity or metric tensor). The Dirichlet problem is well posed under the same regularity and ellipticity bounds. All conclusions of 6.1 and 6.2 hold for direct reconstruction. Note: for inverse problems the anisotropic case has a gauge (boundary fixing diffeomorphisms), but for the direct Dirichlet problem uniqueness and the MH identification F(M) isomorphic to G(boundary M) persist without modification.

D) Mixed boundary conditions.
For Dirichlet on part of the boundary and Neumann or Robin on the complement, well posedness holds under standard compatibility conditions. One adapts G to include data on the Dirichlet portion and the correct functional on the Neumann portion. The same existence, uniqueness, and locality arguments produce MH-type results.

E) Noncompact domains with controlled geometry.
For exterior domains or manifolds with ends, impose decay or radiation conditions. The Poisson operator exists in appropriate weighted spaces, and the sheaf structure is formulated on

relatively compact subsets with a direct limit. Exactness under limits must be checked, but often holds under standard geometric and spectral assumptions.

F) Discrete analogues.
On a finite graph, fix a subset of boundary vertices. For any function on the boundary, there is a unique harmonic extension to the interior minimizing the Dirichlet energy. The map boundary values to harmonic interior is linear, bijective onto Harmonic functions, and satisfies a literal sheaf property with respect to induced subgraphs. This gives a combinatorial MH theorem.

G) Stability and quantitative bounds.
In all variants above, one can state norm estimates of the form norm P(phi) in H^{s} bounded by C times norm phi in H^{s - 1 over 2}, and continuity of eta and its inverse. These quantitative estimates deliver the refinement stability and exactness under limits required by the MH template.

Summary. The Laplace–Dirichlet prototype gives a complete, fully rigorous MH theorem. By importing standard elliptic theory and adapted function spaces, the same structure extends to broad classes of operators and geometries, provided one keeps the admissibility guardrails: uniform ellipticity, suitable boundary regularity, unique continuation, and explicit control of the boundary category and observables.

7. Programmatic Roadmap

7.1 Axiomatize C, D, partial for target domains

Goal. Fix the ambient mathematics so every later claim has unambiguous meaning.

- **Pick your habitat.** Examples:
    - Topology or geometry: compact manifolds with boundary and smooth maps.
    - Algebraic geometry: normal varieties with boundary divisors and proper morphisms.
    - Operator algebras: C star systems with a designated boundary subalgebra and *-homomorphisms.
- **Define the bulk category C.** Specify objects, morphisms, allowed equivalences, and any finiteness or regularity conditions. Record which pathologies are excluded.
- **Define the boundary category D.** Decide what the boundary object is: a boundary manifold, a divisor with scheme structure, a corona algebra, a representation category, or a boundary operator.

- **Define the boundary functor partial from C to D.** It must be functorial on morphisms and compatible with equivalences in C. State exactly when partial is defined and how it behaves under sums, products, and limits.
- **Admissible subcategory.** Declare C adm inside C where reconstruction is expected to hold. Give a short list of admissibility tests a reader can check.

Deliverable: a 1 page spec enumerating objects, morphisms, equivalence relation, and the precise definition of partial.

7.2 Choose F, G so they are local and exact; specify invariants to preserve

Goal. Fix observable functors whose data can be compared across bulk and boundary.

- **Pick targets Alg.** Examples: commutative rings, C star algebras, von Neumann algebras, graded modules, sheaves, or triangulated categories.
- **Define F on bulks and G on boundaries.** Examples:
    - Geometry: F of X equals harmonic functions, cohomology, or solution spaces; G of partial X equals traces, boundary operators, or boundary cohomology.
    - Algebraic geometry: F equals QCoh on X or derived categories; G equals QCoh on the boundary with glueing data.
    - Operator algebraic: F equals the interior observable algebra; G equals the boundary algebra or its multiplier algebra.
- **Locality.** Prove F is a sheaf or cosheaf on a natural site of covers. Prove the analogous property for G on boundary covers. Write the exact equalizer or coequalizer sequences explicitly.
- **Exactness under limits.** Prove that F of limit equals limit of F for the directed systems you will use, and similarly for G with colimits. State the hypotheses that guarantee this, such as compactness or Mittag Leffler.
- **Invariants to preserve.** List the structures the comparison must respect: products, tensor, grading, positivity, order structure, spectra, K theory, or entropy-like functionals.

Deliverable: a 1 page lemma sheet stating locality and exactness of F and G, with references or short proofs.

7.3 Prove injectivity, meaning the boundary distinguishes interiors

Goal. Show that distinct bulks in C adm cannot share the same boundary data in D.

- **Statement.** If G of partial X1 is isomorphic to G of partial X2, then X1 is isomorphic to X2 in C adm.
- **Techniques.**
    - Unique continuation or rigidity: push boundary equalities inward by PDE or analytic continuation.
    - Duality or reconstruction: show an interior invariant is fully readable from G of partial X, e.g., recover the interior algebra, metric, or category.
    - Microlocal or representation separation: prove that G encodes enough directions or tensor data to separate objects.
- **Gauge handling.** Identify any unavoidable equivalence (for example boundary fixing diffeomorphisms) and quotient C by it. Restate injectivity modulo that gauge.
- **Counterexample guard.** Exhibit a failure just outside the hypotheses to demonstrate sharpness.

Deliverable: a theorem with a clean hypothesis list and a short proof sketch indicating the separating mechanism.

7.4 Prove surjectivity, meaning which boundaries are realizable

Goal. Characterize the image of partial, or of the boundary observable functor, and build bulks from admissible boundary data.

- **Realization problem.** For Y in D, decide whether there exists X in C adm with partial X isomorphic to Y and with F of X matching a prescribed element of G of Y.
- **Criteria.** Provide necessary and sufficient boundary conditions: compatibility, positivity, regularity, and normalization constraints. State them as a finite checklist when possible.
- **Construction.** Give a functorial recipe R from D adm to C adm, such as a Poisson extension, welding from boundary homeomorphisms, spectrum construction from an algebra, or Tannaka reconstruction from tensor data.
- **Stability.** Prove continuity of R with respect to natural topologies on boundary data. Quantify stability if possible, even if only logarithmic.

- **Obstructions.** Identify boundary data that cannot arise from any bulk; explain the obstruction in invariant terms.

Deliverable: a theorem identifying the image of partial and a constructive right inverse R on D adm, with a stability corollary.

7.5 Establish equivalence up to isomorphism, or describe the kernel precisely

Goal. Upgrade injectivity plus surjectivity to an equivalence, or else give a sharp kernel description.

- **Equivalence route.**
    - Functors: show partial from C adm to D adm and R from D adm to C adm form an adjoint equivalence with unit and counit natural isomorphisms.
    - Observable compatibility: prove F is naturally isomorphic to G composed with partial, and F composed with R is naturally isomorphic to G on D adm.
    - Monoidal and limit compatibility: check that the unit and counit respect sums, tensor, products, and directed limits.
- **Kernel route.** If full equivalence fails, describe exactly the equivalence relation on C adm that boundary data cannot see. Examples: interior gauge actions, invisible microlocal modes, derived equivalences. Prove that partial factors through the quotient and is faithful and essentially surjective on that quotient.
- **Minimality and maximality.** Show that your admissible classes C adm and D adm are, in a precise sense, as large as possible given injectivity, surjectivity, and stability. Provide counterexamples at the boundary of admissibility.

Deliverable: a main theorem stating either an equivalence of categories between C adm and a reflective or coreflective subcategory of D adm, or a kernel classification theorem that identifies precisely what the boundary forgets.

Outcome. Following 7.1 to 7.5 yields a complete, theorem level realization of mathematical holography in a chosen habitat: clear objects and boundaries, local and exact observables, injectivity and surjectivity with stability, and finally an equivalence or a kernel description that tells the whole truth.

8. Implications for Scholarship and Pedagogy

8.1 Avoiding category errors: conjecture vs theorem in survey writing

- **State status up front.** Write plainly: “The holographic principle is a conjectural organizing idea; we prove only model-specific theorems.”
- **Name the habitat.** Every precise claim should specify the setting (for example compact manifolds with boundary of class C^{1,1}, or commutative C-star algebras).
- **Separate evidence from proof.** Label items as “theorem,” “proposition,” “corollary,” “conjecture,” or “heuristic.” Avoid phrases like “it follows that” when you mean “it suggests that.”
- **Cite the sharp result, not the slogan.** If you use a boundary determines bulk idea, point to the exact theorem you rely on (for example Calderón uniqueness, Radon inversion, Gelfand duality), and list its hypotheses.
- **Acknowledge scope.** Add one line on failures just outside the hypotheses so readers see the boundary of validity.
- **Ban bait-and-switch.** Do not present a special-case theorem and then speak as if a universal principle is proved.

8.2 How to teach “boundary encodes bulk” with real theorems

- **Lead with a provable prototype.** Start from the Dirichlet problem on a compact manifold: trace and Poisson operators give an isomorphism between harmonic functions and boundary data. This is a complete worked example of the MH template.
- **Map to the template.** After the prototype, identify C, D, partial, F, G explicitly so students see how a slogan becomes a diagram and then a proof.
- **Offer a tour of habitats.** Give three short modules:
    1. **Inverse problems**: Dirichlet to Neumann map and uniqueness, plus a one-slide note on stability.
    2. **Integral geometry**: Radon transform, injectivity and filtered backprojection.
    3. **Dualities**: Gelfand and Tannaka–Krein as reconstruction theorems.
- **Show failure modes.** Include one counterexample each for non-uniqueness, instability, and low-regularity pathologies. Students should leave knowing why hypotheses matter.

- **Assessment by construction.** Assign a mini-project: choose a setting, define C, D, partial, propose F and G, and prove at least injectivity or a local gluing statement.
- **Language discipline.** Require precise labels: “We prove MH target T1” instead of “We prove holography.”

8.3 A checklist for reviewers and authors (terminology, hypotheses, claims)

Terminology

- Does the manuscript distinguish “conjecture,” “principle,” and “theorem”?
- Are “bulk,” “boundary,” “observable,” and “reconstruction” defined in the chosen mathematical habitat?
- Are functors and natural maps named and typed, with domains and codomains specified?

Hypotheses

- Is the admissible class C adm stated explicitly (regularity, geometry, finiteness, gauge choices)?
- Are boundary observables specified and shown to be sufficiently rich to separate objects?
- Are locality and exactness addressed: sheaf or cosheaf properties, and compatibility with covers and limits?
- Is stability discussed quantitatively or at least qualitatively?

Claims

- Which MH target is achieved:
  T1 isomorphism of observables,
  T2 injectivity or faithfulness,
  T3 reconstruction via a right inverse and essential surjectivity,
  or a full equivalence of categories?
- Are counterexamples or known obstructions acknowledged just beyond the hypotheses?
- Are monoidal properties and limit behavior verified if used in constructions?
- Is any “principle-level” language tightly confined to discussion sections, not stated as proved results?

Presentation

- Provide a one-page spec of C, D, partial, F, G before main theorems.
- Include a “failure and necessity” box listing which hypothesis each counterexample breaks.
- Summarize contributions with verbs that match proof strength: “establish injectivity,” “characterize image,” “construct right inverse,” “prove equivalence,” rather than “demonstrate holography.”

Bottom line
Good scholarship names the conjecture as such, proves rigorous MH-style theorems in precise habitats, and teaches readers where the bridge from slogan to proof stands firm—and where it does not.

9. FAQ: Clearing Common Misconceptions

9.1 “Isn’t there already a general proof?”

No. There is no single, community standard theorem that formalizes a general bulk to boundary equivalence and proves it from agreed axioms. What exists are rigorous theorems in specific habitats that realize parts of the idea, plus programmatic conjectures. A general proof would require a fixed framework, precise objects and morphisms, explicit hypotheses, and a derivation of a stated equivalence or reconstruction result. Until that package exists, the status is conjectural.

9.2 “Does one example upgrade a principle to a theorem?”

No. An example, even a spectacular one, shows existence, not universality. A theorem needs quantifiers and scope: for all objects X in an admissible class C adm, boundary data G of partial X determine interior data F of X in a specified sense. One example only proves “there exists X,” not “for all X.” Worse, strong examples can mask failure modes; the paper must show injectivity, surjectivity, stability, and locality across the class, not just in a single case.

9.3 “What would falsify an MH conjecture?”

Three kinds of counter evidence:

1. Non uniqueness. Two non isomorphic interiors X1 and X2 in the admissible class with identical boundary observables G of partial X1 equal to G of partial X2.

2. Non realizability. Boundary data Y that pass the stated compatibility conditions but for which no interior X with partial X isomorphic to Y exists.
3. Instability or failure of locality. Reconstruction that is discontinuous beyond the allowed estimates, or boundary to bulk maps that fail to commute with covers or limits as claimed. Any of these, inside the stated hypotheses, falsifies the conjecture or forces a narrower admissible class.

9.4 "How is this different from mere compression?"

Compression is lossy or lossless reduction of data size relative to a chosen encoding. MH asks for something stronger and structural:

- Sufficiency. The boundary observables are sufficient statistics for the interior object within the admissible class.
- Functoriality and locality. The comparison maps are natural and compatible with restriction, gluing, and limits.
- Reconstruction. There is injectivity, and ideally a constructive right inverse from boundary to bulk, not just a shorter description.
- Invariants preserved. Spectra, tensor structure, algebraic operations, or categorical features are respected, not merely approximated.
  In short, MH is about equivalence or canonical reconstruction in a mathematical habitat, whereas compression is about economical encoding under a metric of fidelity. A compressed file that cannot be uniquely decoded within the class is not MH; a boundary dataset that functorially reconstructs the interior is.

10. Conclusion

10.1 The honest status: organizing conjecture, rich theorem fragments

The phrase "holographic principle" functions today as a powerful organizing conjecture rather than a settled theorem. There is no single, community standard statement with agreed hypotheses and proof. What we do have are rigorous, habitat specific theorems that instantiate the boundary encodes bulk pattern: Dirichlet uniqueness and Poisson operators, inverse boundary results of Calderón type, injective and invertible Radon transforms, dualities such as Gelfand and Tannaka–Krein, and conformal welding. These are real theorems with clear scopes, and they collectively motivate a precise template for mathematical holography. The lesson is conceptual hygiene: celebrate what is proved, label what is conjectured, and make the bridge between them explicit.

10.2 The constructive path: from template to precise, provable MH theorems

The MH template turns a slogan into a program. Fix a bulk category, a boundary category, a functor from bulk to boundary, and observable functors that are local and exact. Prove, in this habitat, at least injectivity of the boundary data, then characterize the image, then build a right inverse, and finally upgrade to an equivalence or a sharp kernel description. Throughout, enforce admissibility: regularity, unique continuation, geometric tameness, data richness, and stability. Insist that results commute with covers, limits, and monoidal sums. Each of these steps is standard mathematical work, and each tightens a vague principle into a theorem. The payoff is twofold: new reconstruction results inside established areas, and a shared language that prevents category errors across fields.

10.3 Open problems and suggested benchmarks

- Minimal admissibility. Identify the weakest regularity and geometric hypotheses under which boundary observables remain separating and stable.
- Quantitative stability. Upgrade uniqueness results to sharp continuity or modulus of continuity bounds; aim for optimal exponents in Sobolev or Besov scales.
- Kernel classification. In settings where equivalence fails, describe precisely what the boundary forgets and factor the functor through the correct quotient.
- Monoidal and limit exactness. Prove that reconstruction commutes with disjoint unions, products, and directed refinements beyond classical cases, or exhibit counterexamples.
- New habitats. Formulate and prove MH theorems for stratified spaces, manifolds with corners using adapted function spaces, noncommutative boundaries in operator algebras, or algebraic varieties with singular divisors.
- Microlocal completeness. In integral geometry, characterize directional data sets that suffice for full reconstruction and quantify artifact suppression.
- Categorical strengthening. Replace isomorphisms of sets or algebras by equivalences of categories or higher categories when structure demands it.
- Benchmark problems. Propose canonical test suites:
    1. Elliptic inverse problems on C^{1,1} domains with piecewise analytic coefficients, with target stability rates.
    2. Limited angle Radon inversion with microlocal recovery guarantees.
    3. Reconstruction from boundary tensor data in a Tannaka style setting with specified separating families.

4. Welding for quasisymmetric boundaries with explicit continuity moduli.

A principled literature can now say exactly what “boundary encodes bulk” means in each mathematical habitat, prove it where true, and delimit it where it fails. The conjecture remains an inspiring guide; theorems make it real.

Appendices

A. Formal MH template (copy-safe notation), with categorical diagrams

Objects and functors

- Bulk category: C
- Boundary category: D
- Boundary functor: partial: C -> D
- Observable functors: F: C -> Alg, G: D -> Alg
- Natural comparison map (to be constructed): eta_X: F(X) -> G(partial X)

Locality (sheaf or cosheaf)

- F is a sheaf on a site of C if for any cover {U_i -> X} the equalizer condition holds: F(X) -> product_i F(U_i) ⇒ product_{i,j} F(U_i intersect U_j) is exact at F(X).
- G is a cosheaf on a site of D if for {V_i -> Y} the coequalizer condition holds: coproduct_{i,j} G(V_i intersect V_j) ⇒ coproduct_i G(V_i) -> G(Y) is exact at G(Y).

Commuting square for locality

```
F(X)  --eta_X-->  G(partial X)
 |                 |
 v                 v
lim F(U_i)  -->>  lim G(partial U_i)
```

Here lim means the limit over the cover diagram (equalizer), and the double arrow on the bottom suggests the parallel pair induced by overlaps.

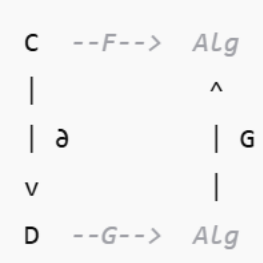
Targets of MH (in increasing strength)

- T1 (isomorphism of observables): eta_X is an isomorphism for all X in C_adm.
- T2 (faithfulness): if G(partial X1) isomorphic to G(partial X2) then X1 isomorphic to X2 in C_adm.
- T3 (reconstruction): there exists R: D_adm -> C_adm with partial o R isomorphic to id on D_adm and a natural isomorphism F isomorphic to G o partial.

Monoidal and limit exactness

- Disjoint unions or products:
  partial(X plus Y) isomorphic to partial X plus partial Y
  F(X plus Y) isomorphic to F(X) tensor F(Y)
  G(partial X plus partial Y) isomorphic to G(partial X) tensor G(partial Y)
  eta respects tensor: eta_{X plus Y} = eta_X tensor eta_Y
- Directed limits and refinements:
  F(lim X_alpha) isomorphic to lim F(X_alpha)
  G(colim Y_beta) isomorphic to colim G(Y_beta)

Category-level picture



Goal: either F is naturally isomorphic to G o partial on C_adm, or partial is an equivalence between C_adm and a reflective or coreflective subcategory of D_adm with R as a quasi-inverse.

B. Summaries of key theorems mapped into MH (hypotheses -> conclusions)

B.1 Elliptic Dirichlet problem (compact smooth manifold with boundary)
Hypotheses: compact smooth M, boundary M nonempty, Laplace–Beltrami Delta.
Conclusion: trace T: Harm(M) -> C infinity(boundary M) is bijective with inverse the Poisson operator P; hence F(M) isomorphic to G(boundary M).
MH tags: C = manifolds, D = boundaries, T1 holds; locality and stability in Sobolev scales.

B.2 Calderón inverse boundary problem (conductivity)
Hypotheses: bounded domain with C^{1,1} boundary; scalar conductivity gamma bounded above and below; appropriate smoothness (dimension dependent).
Conclusion: the Dirichlet-to-Neumann map Lambda_gamma uniquely determines gamma in admissible classes; stability typically logarithmic.
MH tags: T2 (injectivity), partial gamma -> Lambda_gamma, plus constructive reconstruction in special regimes.

B.3 Radon transform (integral geometry)
Hypotheses: functions in Schwartz or suitable Sobolev classes on R^n with full angular coverage.
Conclusion: Radon transform is injective and invertible by filtered backprojection.
MH tags: T3 with explicit right inverse; stability in appropriate norms; limited-angle breaks injectivity.

B.4 Gelfand duality (compact Hausdorff vs commutative C star)
Hypotheses: compact Hausdorff spaces and unital commutative C star algebras.
Conclusion: contravariant equivalence X <-> C(X).
MH tags: strong T3 as category equivalence.

B.5 Tannaka–Krein (compact groups)
Hypotheses: compact group G; tensor category Rep(G) with a faithful fiber functor to finite dimensional vector spaces.
Conclusion: G is reconstructed as automorphisms of the fiber functor; conversely, such categories arise from groups.
MH tags: strong T3 via categorical reconstruction.

B.6 Conformal welding and Riemann mapping
Hypotheses: simply connected planar domains or quasisymmetric boundary data on the circle.
Conclusion: domain determined up to normalization by boundary correspondence; welding reconstructs complementary domains from boundary homeomorphism.
MH tags: T3 up to natural normalization; stability depends on regularity (quasisymmetric control).

C. Failure-mode catalog with minimal counterexamples

C.1 Non-uniqueness via gauge (anisotropic conductivity)
Setup: tensor conductivity gamma, boundary-fixing diffeomorphism phi.
Phenomenon: gamma and pullback by phi share the same Dirichlet-to-Neumann map.
Lesson: need to quotient by gauge or fix representatives.

C.2 Limited-angle tomography
Setup: Radon data only for directions in a cone.
Phenomenon: distinct f give identical data; invisible wavefront sets.
Lesson: require full angular coverage or microlocal conditions.

C.3 Failure of unique continuation
Setup: second-order elliptic operator with very rough coefficients.
Phenomenon: nontrivial solutions vanish on sets near the boundary; boundary traces fail to determine interior.
Lesson: enforce regularity (for example W^{1,n} with bounds) to recover UCP.

C.4 Irregular boundaries
Setup: wildly nonrectifiable boundary.
Phenomenon: trace and Poisson operators fail or lose boundedness; Dirichlet problem ill-posed.
Lesson: restrict to C^{1,1} or adopt weighted spaces and explicit compatibility.

C.5 Inverse-limit pathology
Setup: reconstruct via refinement X_alpha with no compactness or ML condition.
Phenomenon: F(lim X_alpha) not equal to lim F(X_alpha).
Lesson: state exactness hypotheses for limits or work with derived limits explicitly.

C.6 Representation collapse
Setup: group reconstruction from partial data such as character table only.
Phenomenon: nonisomorphic groups share same character table in some families.
Lesson: tensor structure and a fiber functor are essential data.

C.7 Instability
Setup: Calderón reconstruction with noise on boundary data.
Phenomenon: logarithmic modulus of continuity; small noise produces large interior error.
Lesson: include stability class in the theorem, or restrict to regimes with better estimates.

D. Reviewer's checklist for claims of "holography" in pure-math contexts

D.1 Definitions and scope

- Are C, D, partial, F, G explicitly defined with domains and codomains?
- Is the admissible subclass C_adm stated (regularity, geometry, finiteness, gauge)?
- Are boundary observables G(partial X) described precisely?

D.2 Locality and exactness

- Does F satisfy a sheaf equalizer over chosen covers? Proof or citation.
- Does G satisfy a cosheaf coequalizer over boundary covers?
- Do F and G commute with limits and colimits used in the construction?
- Are monoidal properties (disjoint union, tensor) verified where invoked?

D.3 Strength of results

- Which MH target is actually proved:
  T1 isomorphism of observables,
  T2 injectivity,
  T3 reconstruction via a right inverse or categorical equivalence?
- Are stability estimates provided, or at least a stated modulus of continuity?

D.4 Honesty about failure and gauge

- Are known counterexamples acknowledged just outside the hypotheses?
- Is any unavoidable kernel (for example gauge equivalence) identified and factored out?
- If surjectivity fails, is the image characterized or are obstructions stated?

D.5 Presentation discipline

- Are statements labeled theorem, proposition, conjecture, or heuristic appropriately?
- Are hypotheses minimal and motivated by explicit failures?
- Is there a one-page spec of objects, morphisms, functors, and the exact claim?
- Are diagrams and naturality squares included and correct (commuting as claimed)?

D.6 Reproducibility

- If an algorithmic reconstruction is claimed, are input spaces, norms, and regularization choices specified?
- Are the constants and rates in stability estimates explicit or bounded uniformly on C_adm?
- Is there a small library of test examples or counterexamples demonstrating the theorem and its limits?

Bottom line

A claim of “holography” in pure mathematics should read like any serious reconstruction theorem: precise objects, explicit boundary data, locality and exactness, injectivity and surjectivity with stability, and clear acknowledgement of kernels and counterexamples.

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